

Conditional stability for backward parabolic equations with $\text{Log Lip}_t \times \text{Lip}_x$ -coefficients

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Abstract

In this paper we present an improvement of [Math. Ann. **345** (2009), 213–243], where the authors proved a result concerning continuous dependence for backward parabolic operators whose coefficients are Log-Lipschitz in t and C^2 in x . The C^2 regularity with respect to x had to be assumed for technical reasons. Here we remove this assumption, replacing it with Lipschitz-continuity. The main tools in the proof are Littlewood-Paley theory and Bony's paraproduct as well as a result of Coifman and Meyer [Astérisque **57**, 1978, Th. 35].

Keywords: Conditional stability, backward parabolic equation, low regularity coefficients, weighted energy estimate, Bony's paraproduct

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1. Introduction

In this paper we study the continuous dependence of solutions to the Cauchy problem for a backward parabolic operator, namely

$$Pu = \partial_t u + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u) = 0 \quad (1.1)$$

on the strip $[0, T] \times \mathbb{R}_x^n$ with data

$$u(0, x) = u_0(x) \quad (1.2)$$

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in \mathbb{R}_x^n . The coefficients are supposed to be measurable, real valued and bounded. The matrix $(a_{jk})_{j,k=1,\dots,n}$ is symmetric and positive definite, i.e. there exists $\kappa > 0$ such that

$$\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \geq \kappa |\xi|^2, \quad (1.3)$$

for all $\xi \in \mathbb{R}_\xi^n$.

It is well known that the Cauchy problem (1.1), (1.2) it is not well-posed in the sense of Hadamard [10], [11]. On the one side the smoothing effect of the parabolic operator prevent to have existence results in any reasonable function space and on the other side (relatively) elementary examples show that also uniqueness is not valid without additional assumptions on the solutions and on the operator (see [18]; for a more precise discussion on uniqueness of the solutions to the Cauchy problem for a backward parabolic equation we quote the papers [16], [14], [5], [7], [8]).

In the celebrated paper [13], John introduced the notion of well-behaved problem in which also not well-posed problem can be included: roughly speaking a problem is well-behaved if its solutions in a space \mathcal{H} depend continuously on the data belonging to a space \mathcal{K} , provided they satisfy a prescribed bound in possibly another space \mathcal{H}' .

The well-behavedness for (1.1), (1.2) in the space

$$\mathcal{H} = C^0([0, T], L^2(\mathbb{R}_x^n)) \cap C^0([0, T], H^1(\mathbb{R}_x^n)) \cap C^1([0, T], L^2(\mathbb{R}_x^n)) \quad (1.4)$$

with continuous dependence with respect to the data in $L^2(\mathbb{R}_x^n)$, can be deduced from the so called logarithmic convexity of the norm of the solutions to (1.1), as proved by Agmon and Nirenberg in [1]. A similar result was obtained by Glagoleva in [9] and in a more precise and general form by Hurd in [12]. Hurd's result can be summarized as follows:

suppose that the coefficients a_{jk} are Lipschitz-continuous; for every $T' \in (0, T)$ and $D > 0$, there exist $\rho > 0$, $\delta \in (0, 1)$ and $M > 0$ such that if $u \in \mathcal{H}$ (\mathcal{H} defined in (1.4)), is a solution of $Pu \equiv 0$ on $[0, T] \times \mathbb{R}_x^n$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(T, \cdot)\|_{L^2} \leq D$, then

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M \|u(0, \cdot)\|_{L^2}^\delta, \quad (1.5)$$

where the constants ρ , M and δ depend only on T' , D , the ellipticity constant of P and the Lipschitz constant of the coefficients with respect to t .

Hurd's proof relies on some (complicate) energy estimates and it turns out that Lipschitz-continuity of the coefficients a_{jk} is an essential requirement.

In the present paper we are interested in relaxing the regularity hypothesis on the coefficients a_{jk} . Our starting point are the results contained in [6]. In that paper an example showed that if the coefficients a_{jk} are not Lipschitz-continuous in time then the estimate (1.5) does not hold in general, and if the coefficients are log-Lipschitz-continuous in time then a condition weaker than

(1.5) is valid. However, in order to obtain this weaker estimate, a technical difficulty imposed to assume the C^2 -regularity with respect to the space variables. Here we overcome this point and we remove this supplementary and not natural requirement. Our result is the following:

suppose that the coefficients a_{jk} are Lipschitz-continuous with respect to x and log-Lipschitz-continuous with respect to t ; for every $T' \in (0, T)$, $D > 0$ and $s \in (0, 1)$, there exist $\rho > 0$, $\delta \in (0, 1)$ and $M, N > 0$ such that if $u \in \mathcal{H}$ is a solution of $Pu \equiv 0$ on $[0, T] \times \mathbb{R}_x^n$ with $\|u(0, \cdot)\|_{L^2} \leq \rho$ and $\|u(T, \cdot)\|_{L^2} \leq D$, then

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M \exp(-N |\log(\|u(0, \cdot)\|_{H^{-s}})|^\delta),$$

where the constants ρ , M , N and δ depend only on T' , D , s , the ellipticity constant of P , the Lipschitz constant of the coefficients with respect to x and the Log-Lipschitz constant of the coefficients with respect to t .

The main tool in proving this statement is Bony's paraproduct (see [15]) and a theorem by Coifman and Meyer [3, Th. 35], which makes the estimate of a commutator more effective.

Outline of the content. In Sub-Section 2.2, we state our main theorems and make some remarks regarding the comparison with the results of [6].

In Sub-Section 3.1, we present elements of the Littlewood-Paley theory and we develop the necessary machinery of Bony's paraproduct for our proof. After that we proof auxiliary estimates that will be crucial for the proof of our weighted energy estimate in Sub-Sections 3.3 and 3.4. Some proofs are shifted to the appendix in order to make the main results easier to read.

In Section 4, we prove a weighted energy estimate for solutions of (2.1) from which the stability result Theorem 2.4 follows. The derivation of the stability result from the weighted energy estimate is shown in Section 5.

2. Results

2.1. Notation

We consider the backward-parabolic equation

$$Pu = \partial_t u + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} u) = 0 \quad (2.1)$$

on the strip $[0, T] \times \mathbb{R}_x^n$. We suppose that

- for all $(t, x) \in [0, T] \times \mathbb{R}_x^n$ and for all $j, k = 1, \dots, n$,

$$a_{jk}(t, x) = a_{kj}(t, x);$$

- there exists an $\kappa \in (0, 1)$ such that for all $(t, x, \xi) \in [0, T] \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n$,

$$\kappa |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \leq \frac{1}{\kappa} |\xi|^2; \quad (2.2)$$

- for all $j, k = 1, \dots, n$, $a_{jk} \in \text{Log Lip}([0, T], L^\infty(\mathbb{R}_x^n)) \cap L^\infty([0, T], \text{Lip}(\mathbb{R}_x^n))$.

We set

$$A_{LL} := \sup \left\{ \frac{|a_{jk}(t, x) - a_{jk}(s, x)|}{|t - s| |1 + \log |t - s||} : j, k = 1, \dots, n, \right. \\ \left. (t, s, x) \in [0, T]^2 \times \mathbb{R}_x^n, 0 < |s - t| \leq T \right\},$$

$$A := \sup \{ \|\partial_x^\alpha a_{jk}(t, \cdot)\|_{L^\infty} : |\alpha| \leq 1, t \in [0, T] \}.$$

Remark 2.1. If one would like to include lower order terms in (2.1), one has to suppose that those are L^∞ with respect t and also Lip with respect to x . The constants will then additionally depend on constants B and C similarly defined to A .

Remark 2.2. We will often use a letter, say C , to denote a generic numerical constant; and different appearances of the letter C will not necessarily denote the same numerical constant, even in the same line of text. When a constant actually depends on one of the parameters of the problem, it shall be indicate by an index. Sometimes it might be necessary to differentiate between constants so that we will count them with an upper index.

2.2. Main results - stability and weighted energy estimates

We denote by

$$\mathcal{H} := C^0([0, T], L^2(\mathbb{R}_x^n)) \cap C^0([0, T], H^1(\mathbb{R}_x^n)) \cap C^1([0, T], L^2(\mathbb{R}_x^n))$$

the space of solutions of (2.1) for which we prove the stability result.

First we restate the precise local stability result of [6]; we also want to compare the two estimates in the sequel. Keep in mind that in this case the constant A also contains the L^∞ norms of the second spatial derivative of the principal part coefficients.

Theorem 2.3. (Th. 1 in [6]) *There exist a positive constant α_1 and, setting $\sigma := \min\{T, \frac{1}{\alpha_1}\}$, $\bar{\sigma} = \frac{\sigma}{8}$, there exist constants ρ , δ , M and N , such that, whenever $u \in \mathcal{H}$ is a solution of (2.1) with $\|u(0, \cdot)\|_{L^2} \leq \rho$, the inequality*

$$\sup_{t \in [0, \bar{\sigma}]} \|u(t, \cdot)\|_{L^2} \leq M(1 + \|u(\sigma, \cdot)\|_{L^2}) \exp(-N(|\log(\|u(0, \cdot)\|_{L^2})|^\delta))$$

holds true.

The constant α_1 depends only on A_{LL} , A , κ and n , while the constants ρ , δ , M and N depend on A_{LL} , A , κ , n and T .

Let us stress again that the constants α_1 , ρ , δ , M , N depend also on similar constants B and C if one considers also lower order terms. See Remark 2.1.

The next results improves Theorem 2.3: now the principal part coefficients are only Lipschitz continuous with respect to x .

Theorem 2.4. Conditional stability (local). *Let $s \in (0, 1)$. There exist a positive constant α_1 and, setting $\sigma := \min\{T, \frac{1-s}{\alpha_1}\}$, $\bar{\sigma} = \frac{\sigma}{8}$, there exist constants ρ, δ, M and N , such that, whenever $u \in \mathcal{H}$ is a solution of (2.1) with $\|u(0, \cdot)\|_{H^{-s}} \leq \rho$, the inequality*

$$\begin{aligned} & \sup_{t \in [0, \bar{\sigma}]} \|u(t, \cdot)\|_{L^2} \\ & \leq M \left(1 + \frac{1}{\sigma} \max_{t \in [\frac{5}{8}\sigma, \frac{7}{8}\sigma]} \|u(t, \cdot)\|_{L^2} \right) \exp(-N(|\log(\|u(0, \cdot)\|_{H^{-s}})|^\delta)) \end{aligned} \quad (2.3)$$

holds true.

The constant α_1 depends only on A_{LL} , A , κ , s and n , while the constants ρ, δ, M and N depend on A_{LL} , A , κ , s , n and T .

Iterating the local result of Theorem 2.4 a finite number of times, one obtains the following global continuous dependence result.

Theorem 2.5. Conditional stability (global). *Let $s \in (0, 1)$. Then, for $T' \in (0, T)$ and $D > 0$ there exist positive constants ρ', δ', M' and N' , depending only on A_{LL} , A , κ , n , s and T' such that if $u \in \mathcal{H}$ is a solution of (2.1) satisfying $\|u(0, \cdot)\|_{H^{-s}} \leq \rho$ and $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq D$, the inequality*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M' \exp(-N' |\log(\|u(0, \cdot)\|_{H^{-s}})|^{\delta'})$$

holds true.

Remark 2.6. Theorems 2.4 and 2.5 hold also if one considers equation (2.1) with lower order terms. As already mentioned, one has to assume Lipschitz-regularity in x and the additional dependence of the constants on the L^∞ -norm and the Lip-norm of those coefficients.

2.2.1. Weighted energy estimates

The proof of Theorem 2.4 relies on an appropriate weighted energy estimate. The choice of the weight function is connected with the modulus of continuity with respect to t as in [6]. A similar situation occurred in [5], [7], where backward-uniqueness for parabolic operators by means of suitable Carleman estimates was obtained. In both cases, the weight function was deduced as a solution of a second order non-linear ordinary differential equation.

Let us now introduce the weight function that we are going to use here. For $s > 0$, let $\mu(s) = s(1 + |\log(s)|)$. For $\tau \geq 1$, we define

$$\theta(\tau) := \int_{\frac{1}{\tau}}^1 \frac{1}{\mu(s)} ds = \log(1 + |\log(\tau)|).$$

The function $\theta : [1, +\infty) \rightarrow [0, +\infty)$ is bijective and strictly increasing. For $y \in (0, 1]$ and $\lambda > 1$, we set $\psi_\lambda(y) = \theta^{-1}(-\lambda \log(y)) = \exp(y^{-\lambda} - 1)$ and we define

$$\Phi_\lambda(y) := - \int_y^1 \psi_\lambda(z) dz.$$

The function $\Phi_\lambda : (0, 1] \rightarrow (-\infty, 0]$ is bijective and strictly increasing; moreover, it satisfies

$$y\Phi_\lambda''(y) = -\lambda(\Phi_\lambda'(y))^2\mu\left(\frac{1}{\Phi_\lambda'(y)}\right) = -\lambda\Phi_\lambda'(y)(1 + |\log\left(\frac{1}{\Phi_\lambda'(y)}\right)|). \quad (2.4)$$

This is the second order differential equation we mentioned above. The reason for this choice is made clear in [6, Sec. 2]. The computations in [5], [7] lead to a different differential equation and consequently to a different weight. In the next lemma, we collect some properties of the functions ψ_λ and Φ_λ . The proof is left to the reader.

Lemma 2.7. *Let $\zeta > 1$. Then, for $y \leq 1/\zeta$,*

$$\psi_\lambda(\zeta y) = \exp(\zeta^{-\lambda} - 1)(\psi_\lambda(y))^{\zeta^{-\lambda}}.$$

Define $\Lambda_\lambda(y) := y\Phi_\lambda(1/y)$. Then the function $\Lambda_\lambda : [1, +\infty) \rightarrow (-\infty, 0]$ is bijective and

$$\lim_{z \rightarrow -\infty} -\frac{1}{z}\psi_\lambda\left(\frac{1}{\Lambda_\lambda^{-1}(z)}\right) = +\infty.$$

With these preparations, we are ready to state the energy estimate which will be needed to prove Theorem 2.4.

Proposition 2.8. Weighted energy estimate. *Let $s \in (0, 1)$. Then, there exist positive constants $\bar{\lambda} > 1$, $\bar{\gamma}$, α_1 and $M > 0$ such that, setting $\alpha := \max\{\alpha_1, T^{-1}\}$, $\sigma := \frac{1-s}{\alpha}$, $\tau := \frac{\sigma}{4}$, letting $\beta \geq \sigma + \tau$ be a free parameter, then, for $u \in \mathcal{H}$ a solution of equation (2.1), one has*

$$\begin{aligned} \int_0^p e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-s-\alpha t}}^2 dt \\ \leq M \left((p + \tau) e^{2\gamma p} e^{-2\beta\Phi_\lambda(\frac{p+\tau}{\beta})} \|u(p, \cdot)\|_{H^{1-s-\alpha p}}^2 \right. \\ \left. + \tau \Phi_\lambda' \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right) \end{aligned} \quad (2.5)$$

for all $p \in [0, \frac{7}{8}\sigma]$, $\lambda \geq \bar{\lambda}$ and $\gamma \geq \bar{\gamma}$. The constant α_1 depends only on A_{LL} , A , κ , s and n , while the constants $\bar{\lambda}$, $\bar{\gamma}$ and M depend on A_{LL} , A , κ , s , n and T .

Notice that this energy inequality undergoes a loss of derivatives. This phenomenon also occurred in [2], [4] in the context of hyperbolic equations with Log-Lipschitz coefficients.

3. Littlewood-Paley theory and Bony's paraproduct

In this section, we review some elements of the Littlewood-Paley decomposition which we shall use throughout this paper to define Bony's paraproduct. The proofs which are not contained in this section can be found in [6], [7] and also [15].

3.1. Littlewood-Paley decomposition

Let $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi(s) \leq 1$ be an even function and such that $\chi(s) = 1$ for $|s| \leq 11/10$ and $\chi(s) = 0$ for $|s| \geq 19/10$. We now define $\chi_k(\xi) = \chi(2^{-k}|\xi|)$ for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}_x^n$. Denoting by \mathcal{F} the Fourier-transform $x \rightarrow \xi$ and by \mathcal{F}^{-1} its inverse, we define the operators

$$\begin{aligned} S_{-1}u &= 0 \quad \text{and} \quad S_k u = \chi_k(D_x)u = \mathcal{F}^{-1}(\chi_k(\cdot)\mathcal{F}(u)(\cdot)), \quad k \geq 0, \\ \Delta_0 u &= S_0 u \quad \text{and} \quad \Delta_k u = S_k u - S_{k-1} u, \quad k \geq 1. \end{aligned}$$

We define

$$\text{spec}(u) := \text{supp}(\mathcal{F}(u))$$

and we will use the abbreviation $\Delta_k u = u_k$. For $u \in \mathcal{S}'(\mathbb{R}_x^n)$, we have

$$u = \lim_{k \rightarrow +\infty} S_k u = \sum_{k \geq 0} \Delta_k u$$

in the sense of $\mathcal{S}'(\mathbb{R}_x^n)$.

We shall make use of the classical

Proposition 3.1 (Bernstein's inequalities). *Let $u \in \mathcal{S}'(\mathbb{R}_x^n)$. Then, for $\nu \geq 1$,*

$$2^{\nu-1} \|u_\nu\|_{L^2} \leq \|\nabla_x u_\nu\|_{L^2} \leq 2^{\nu+1} \|u_\nu\|_{L^2}. \quad (3.1)$$

The right inequality of (3.1) holds also for $\nu = 0$.

In the following two propositions, we recall the characterization of the classical Sobolev spaces and Lipschitz-continuous functions via Littlewood-Paley decomposition.

Proposition 3.2. *Let $s \in \mathbb{R}$. Then, a tempered distribution $u \in \mathcal{S}'(\mathbb{R}_x^n)$ belongs to $H^s(\mathbb{R}_x^n)$ iff the following two conditions hold:*

- (i) *for all $k \geq 0$, $\Delta_k u \in L^2(\mathbb{R}_x^n)$*
- (ii) *the sequence $\{\delta_k\}_{k \in \mathbb{N}}$, where $\delta_k := 2^{ks} \|\Delta_k u\|_{L^2}$, belongs to $l^2(\mathbb{N})$.*

Moreover, there exists $C_s \geq 1$ such that for all $u \in H^s(\mathbb{R}_x^n)$, we have

$$\frac{1}{C_s} \|u\|_{H^s} \leq \|\{\delta_k\}_k\|_{l^2} \leq C_s \|u\|_{H^s}.$$

Proposition 3.3. *Let $s \in \mathbb{R}$ and $R > 2$. If a sequence $\{u_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}_x^n)$ satisfies*

- (i) *$\text{spec}(u_0) \subseteq \{|\xi| \leq R\}$ and $\text{spec}(u_k) \subseteq \{R^{-1}2^k \leq |\xi| \leq 2R2^k\}$ for all $k \geq 1$ and*
- (ii) *the sequence $\{\delta_k\}_{k \in \mathbb{N}}$, where $\delta_k := 2^{ks} \|u_k\|_{L^2}$, belongs to $l^2(\mathbb{N})$*

then $u = \sum_{k \geq 0} u_k \in H^s(\mathbb{R}_x^n)$ and there exists $C_s \geq 1$ such that we have

$$\frac{1}{C_s} \|u\|_{H^s} \leq \|\{\delta_k\}_k\|_{l^2} \leq C_s \|u\|_{H^s}.$$

When $s > 0$, it is enough to assume that for all $k \geq 0$,

$$\text{spec}(u_k) \subseteq \{|\xi| \leq R2^k\}$$

holds true.

Proposition 3.4. *A function $a \in L^\infty(\mathbb{R}_x^n)$ belongs to $\text{Lip}(\mathbb{R}_x^n)$ iff*

$$\sup_{k \in \mathbb{N}_0} \|\nabla_x(S_k a)\|_{L^\infty} < +\infty.$$

Moreover, there exists a positive constant C such that if $a \in \text{Lip}(\mathbb{R}_x^n)$, then

$$\|\Delta_k a\|_{L^\infty} \leq C2^{-k} \|a\|_{\text{Lip}}, \quad \text{and} \quad \|\nabla_x(S_k a)\|_{L^\infty} \leq C \|a\|_{\text{Lip}}.$$

3.2. Bony's (modified) paraproduct

Let $a \in L^\infty(\mathbb{R}_x^n)$. Then, Bony's paraproduct of a and $u \in H^s(\mathbb{R}_x^n)$ is defined as

$$T_a u = \sum_{k \geq 3} S_{k-3} a \Delta_k u.$$

For the proof of our continuous dependence result it is essential that T_a is a positive operator. Unfortunately, this is not implied by $a(x) \geq \kappa > 0$. Therefore, we have to modify the paraproduct a little bit. We introduce the operator

$$T_a^m u = S_{m-1} a S_{m+2} u + \sum_{k \geq m+3} S_{k-3} a \Delta_k u, \quad (3.2)$$

where $m \in \mathbb{N}_0$; note $T_a^0 = T_a$. As shall be shown in one of the subsequent propositions, the operator T_a^m is a positive operator for positive a provided that m is sufficiently large. The proofs of the subsequent propositions can be found in [7]. We give proofs only if there are important points to them which are not contained in [7].

Proposition 3.5. *Let $m \in \mathbb{N}$, $s \in \mathbb{R}$ and $a \in L^\infty(\mathbb{R}_x^n)$. Then, T_a^m maps $H^s(\mathbb{R}_x^n)$ continuously into $H^s(\mathbb{R}_x^n)$, i.e. there exists a constant $C_{m,s} > 0$ such that*

$$\|T_a^m u\|_{H^s} \leq C_{m,s} \|a\|_{L^\infty} \|u\|_{H^s}.$$

If $m \in \mathbb{N}_{\geq 3}$, $s \in (0, 1)$ and $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ then $a - T_a^m$ maps $H^{-s}(\mathbb{R}_x^n)$ continuously into $H^{1-s}(\mathbb{R}_x^n)$, i.e. there exists a constant $C_{m,s} > 0$ such that

$$\|au - T_a^m u\|_{H^{1-s}} \leq C_{m,s} \|a\|_{\text{Lip}} \|u\|_{H^{-s}}.$$

$C_{m,s}$ is independent of s if s is chosen in a compact subset of $(0, 1)$.

Next we state a positivity result for T_a^m .

Proposition 3.6. *Let $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and suppose that $a(x) \geq \kappa > 0$ for all $x \in \mathbb{R}_x^n$. Then, there exists a constant $m_0 = m_0(\kappa, \|a\|_{\text{Lip}})$ such that*

$$\text{Re} \langle T_a^m u | u \rangle_{L^2} \geq \frac{\kappa}{2} \|u\|_{L^2}^2,$$

for all $u \in L^2(\mathbb{R}_x^n)$ and $m \geq m_0$. A similar result is true for vector-valued functions if a is replaced by a positive symmetric matrix.

The next proposition is needed since T_a^m is not self-adjoint. However, the operator $(T_a^m - (T_a^m)^*)\partial_{x_j}$ is of order 0 and maps, if a is Lipschitz, L^2 continuously into L^2 .

Proposition 3.7. *Let $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and $u \in L^2(\mathbb{R}_x^n)$. Then, there exists a constant $C_m > 0$ such that*

$$\|(T_a^m - (T_a^m)^*)\partial_{x_j} u\|_{L^2} \leq C_m \|a\|_{\text{Lip}(\mathbb{R}^n)} \|u\|_{L^2}.$$

3.3. Auxiliary estimates for $a - T_a^m$

Let $m \geq 3$. We set

$$(a - T_a^m)w = \sum_{k \geq m} \Delta_k a S_{k-3} w + \sum_{k \geq m} \left(\sum_{\substack{j \geq 0 \\ |j-k| \leq 2}} \Delta_k a \Delta_j w \right) = \Omega_1 w + \Omega_2 w. \quad (3.3)$$

For our proof of the energy estimate from which we derive the stability result we need some estimates for terms involving $\Delta_\nu((a - T_a^m)w)$. To handle these terms, we introduce a second Littlewood-Paley decomposition depending on a parameter μ and we look at $\sum_{\mu \geq 0} \Delta_\nu((a - T_a^m)w_\mu)$. To derive estimates for those terms we need appropriate estimates for $\Delta_\nu \Omega_1 w_\mu$ and $\Delta_\nu \Omega_2 w_\mu$. Let us first analyze the spectra of $\Delta_\nu \Omega_1 w$ and $\Delta_\nu \Omega_2 w$: From the definition of S_k and Δ_k in Section 3.1 we see that

$$\text{spec}(\Delta_k a S_{k-3} w) \subseteq \{2^{k-2} \leq |\xi| \leq 2^{k+2}\}$$

and therefore

$$\Delta_\nu \Omega_1 w = \sum_{\substack{k \geq m \\ |k-\nu| \leq 2}} \Delta_\nu(\Delta_k a S_{k-3} w)$$

since $\Delta_\nu(\Delta_k a S_{k-3} w) \equiv 0$ for $|\nu - k| \geq 3$. Replacing now w by w_μ we get

$$\text{spec}(S_{k-3} w_\mu) \subseteq \begin{cases} \emptyset & : k \leq \mu + 1, \\ \{2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\} & : k \geq \mu + 2, \end{cases}$$

and obtain from this

$$\text{spec}(\Delta_k a S_{k-3} w_\mu) \subseteq \begin{cases} \emptyset & : k \leq \mu + 1, \\ \{|\xi| \leq 2^{k+2}\} & : k = \mu + 2, \\ \{2^{k-2} \leq |\xi| \leq 2^{k+2}\} & : k \geq \mu + 2. \end{cases}$$

With this we get

$$\Delta_\nu \Omega_1 w_\mu = \sum_{\substack{k \geq \max\{m, \mu+2\} \\ |\nu-k| \leq 1}} \Delta_\nu(\Delta_k a S_{k-3} w_\mu).$$

Further, we also get $\Delta_\nu \Omega_1 w_\mu \equiv 0$ for $\nu \leq \mu - 1$. Now we look at $\Delta_\nu \Omega_2 w_\mu$. We have

$$\begin{aligned} \Delta_\nu \Omega_2 w_\mu &= \Delta_\nu \left(\sum_{k \geq m} \left(\sum_{\substack{j \geq 0 \\ |j-k| \leq 2}} \Delta_k a \Delta_j w_\mu \right) \right) \\ &= \Delta_\nu \left(\sum_{|\mu-j| \leq 1} \sum_{|j-k| \leq 2} \Delta_k a \Delta_j w_\mu \right) \end{aligned}$$

since

$$\text{spec}(\Delta_j(\Delta_\mu w)) \subseteq \begin{cases} \emptyset & : |j - \mu| \geq 2, \\ \{2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\} & : |j - \mu| \leq 1. \end{cases}$$

From that we get

$$\Delta_\nu \Omega_2 w_\mu = \Delta_\nu \left(\sum_{|\mu-j| \leq 1} \sum_{\substack{k \geq m \\ |k-j| \leq 2}} \Delta_k a \Delta_j(\Delta_\mu w) \right) \quad (3.4)$$

with

$$\text{spec}(\Delta_\nu \Omega_2 w_\mu) \subseteq \{|\xi| \leq 2^{\mu+5}\}, \quad \nu \leq \mu + 5.$$

For all $\nu \geq \mu + 6$ we have $\Delta_\nu \Omega_2 w_\mu \equiv 0$.

We prove now some technical lemmas which we will use later on.

Lemma 3.8. *Let $s' \in (0, 1)$, $m \in \mathbb{N}_0$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and $w \in L^2(\mathbb{R}_x^n)$. Then, there exist a constant $C > 0$ and a sequence $\{c_\nu^{(\mu)}\}_{\nu \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$, depending on $\Delta_\mu w$, with $\|\{c_\nu^{(\mu)}\}_\nu\|_{l^2} \leq 1$ for all $\mu \geq 0$, such that*

$$\|\Delta_\nu \Omega_1 w_\mu\|_{L^2} \leq C 2^{-\nu(1-s')} c_\nu^{(\mu)} \|w_\mu\|_{L^2}. \quad (3.5)$$

Proof. From our considerations above we have that

$$\Delta_\nu \Omega_1 w_\mu = \sum_{|k-\nu| \leq 2} \Delta_\nu(\Delta_k a S_{k-3} w_\mu)$$

and therefore

$$\begin{aligned}
\|\Delta_\nu(\Omega_1 w_\mu)\|_{L^2} &\leq \sum_{k:|k-\nu|\leq 1} \|\Delta_k a S_{k-3} w_\mu\|_{L^2} \\
&\leq \sum_{|k-\nu|\leq 2} \|\Delta_k a\|_{L^\infty} \|S_{k-3} w_\mu\|_{L^2} \\
&\leq C \sum_{|k-\nu|\leq 2} \|a\|_{\text{Lip}} 2^{-k} \sum_{j\leq k} \|\Delta_j w_\mu\|_{L^2} \\
&= C \|a\|_{\text{Lip}} \sum_{|k-\nu|\leq 2} 2^{-k} \sum_{j\leq k} 2^{ks'} 2^{-ks'} 2^{js'} \underbrace{2^{-js'} \|\Delta_j w_\mu\|_{L^2}}_{=:\varepsilon_j^{(\mu)}} \\
&\leq C \|a\|_{\text{Lip}} \sum_{|k-\nu|\leq 2} 2^{-(1-s')k} \underbrace{\sum_{j\leq k} 2^{-(k-j)s'} \varepsilon_j^{(\mu)}}_{=:f_k^{(\mu)}} \\
&= C \sum_{|k-\nu|\leq 2} \|a\|_{\text{Lip}} 2^{-(1-s')k} f_k^{(\mu)} \\
&\leq C \|a\|_{\text{Lip}} 2^{-(1-s')\nu} \sum_{|k-\nu|\leq 2} f_k^{(\mu)}, \tag{3.6}
\end{aligned}$$

where $\{\varepsilon_j^{(\mu)}\}_{j\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$ with $\|\{\varepsilon_j^{(\mu)}\}\|_{l^2} \approx \|w_\mu\|_{H^{-s'}}$; see Proposition 3.2. The sequence $\{f_k^{(\mu)}\}_{k\in\mathbb{N}_0}$ is a convolution of the sequences $\{\varepsilon_j^{(\mu)}\}_{j\in\mathbb{N}_0}$ and $d_k := 2^{-ks'}$. Using Young's inequality, we obtain

$$\|\{f_k^{(\mu)}\}_k\|_{l^2} = \|\{\{\varepsilon_j^{(\mu)}\} *_{(j)} \{d_k\}\}_k\|_{l^2} \leq \|\{d_k\}_k\|_{l^1} \|\{\varepsilon_j^{(\mu)}\}_j\|_{l^2}.$$

From the formula of the geometric series and the integral criterion, we obtain

$$\|\{d_k\}_k\|_{l^1} \leq \frac{1}{1-2^{-s'}} \leq \frac{C}{s'}$$

and hence, we get

$$\|\{f_k\}_k\|_{l^2} \leq \frac{C}{s'} \|w_\mu\|_{H^{-s'}}.$$

We define

$$c_\nu := \frac{f_{\nu-2}^{(\mu)} + f_{\nu-1}^{(\mu)} + f_\nu^{(\mu)} + f_{\nu+1}^{(\mu)} + f_{\nu+2}^{(\mu)}}{C_{s'} \|w_\mu\|_{H^{-s'}}},$$

where $C_{s'}$ can be chosen such that $\sum_{\nu\geq 0} (c_\nu^{(\mu)})^2 \leq 1$. With this, we get from (3.6)

$$\|\Delta_\nu \Omega_1 w_\mu\|_{L^2} \leq C \|a\|_{\text{Lip}} 2^{-(1-s')\nu} c_\nu^{(\mu)} \|w_\mu\|_{H^{-s'}}.$$

Using the embedding of L^2 into $H^{-s'}$, we finally obtain (3.5). \square

The next lemma deals with the estimate of $\Delta_\nu \Omega_2 w$.

Lemma 3.9. *Let $s' \in (0, 1)$, $m \in \mathbb{N}_0$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and $w \in L^2(\mathbb{R}_x^n)$. Then, there exist a constant $C > 0$ and a sequence $\{\tilde{c}_\nu^{(\mu)}\}_{\nu \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$, depending on $\Delta_\mu w$, with $\|\{\tilde{c}_\nu^{(\mu)}\}_\nu\|_{l^2} \leq 1$ for all $\mu \geq 1$, such that*

$$\|\Delta_\nu \Omega_2 w_\mu\|_{L^2} \leq C \|a\|_{\text{Lip}} \tilde{c}_\nu^{(\mu)} 2^{-\mu} \|w_\mu\|_{L^2}.$$

Proof. Straightforward computations on (3.4) show that $\Omega_2 w \in L^2(\mathbb{R}_x^n)$ if $w \in L^2(\mathbb{R}_x^n)$. Hence, there exists a sequence $\{c_\nu^{(\mu)}\}_{\nu \in \mathbb{N}_0}$, depending on w_μ , with $\|\{c_\nu^{(\mu)}\}_\nu\|_{l^2} \approx \|\Omega_2 w_\mu\|_{L^2}$. From (3.4), we obtain

$$\begin{aligned} \|\Delta_\nu \Omega_2 w_\mu\|_{L^2} &\leq \tilde{c}_\nu^{(\mu)} \|\Omega_2 w_\mu\|_{L^2} \\ &\leq \tilde{c}_\nu^{(\mu)} \sum_{|\mu-j| \leq 1} \sum_{\substack{k \geq m \\ |k-j| \leq 2}} \|\Delta_k a \Delta_j (\Delta_\mu w)\|_{L^2} \\ &\leq \tilde{c}_\nu^{(\mu)} \sum_{|j-\mu| \leq 1} \sum_{\substack{k \geq m \\ |j-k| \leq 2}} 2^{-k} \|a\|_{\text{Lip}} \|w_\mu\|_{L^2} \\ &\leq \|a\|_{\text{Lip}} \tilde{c}_\nu^{(\mu)} 2^{-\mu} \|w_\mu\|_{L^2}, \end{aligned}$$

where $\tilde{c}_\nu^{(\mu)} = c_\nu^{(\mu)} / \|\Omega_2 w_\mu\|_{L^2}$. By construction we have $\sum_{\nu \geq 0} (\tilde{c}_\nu^{(\mu)})^2 \leq 1$ for all $\mu \geq 0$. This concludes the proof. \square

The next proposition is at the very heart of our proof and contains information about the behavior of the Littlewood-Paley pieces of $(a - T_a)w$.

Proposition 3.10. *Let $s \in (0, 1)$, $m \in \mathbb{N}_0$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$, $\alpha > 0$ and $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Then there exists a constant $C_s > 0$ such that, for all $w \in \mathcal{H}$, we have*

$$\begin{aligned} &\sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \langle \partial_{x_i} \partial_t v_\nu(t, \cdot) | \Delta_\nu ((a - T_a^m) \partial_{x_j} w(t, \cdot)) \rangle_{L^2} \\ &\leq \frac{1}{N} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + C_s N \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \end{aligned}$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

The proof of this proposition can be found in the appendix. Following the same ideas one can also prove

Proposition 3.11. *Let $s \in (0, 1)$, $m \in \mathbb{N}_0$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$, $\alpha > 0$ and $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Then there exists a constant $C_s > 0$ such that, for all $w \in \mathcal{H}$, we have*

$$\sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \nu \langle \partial_{x_i} v_\nu(t, \cdot) | \Delta_\nu ((a - T_a^m) \partial_{x_j} w(t, \cdot)) \rangle_{L^2} \leq C_s \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

3.4. Auxiliary estimates for $[\Delta_\nu, T_a^m]$

The next result about commutation will also be crucial in our proof of the energy estimate (2.5). It also plays an essential role in the proof of Carleman estimates for (1.1) with low-regular coefficients [7?] and in the well-posedness for hyperbolic equations with low-regular coefficients [4].

Proposition 3.12. *Let $m \in \mathbb{N}_{\geq 3}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and $s \in (0, 1)$. Then, for $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$ there exists a constant $C_m > 0$ such that for all $w \in \mathcal{H}$*

$$\begin{aligned} & \sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \langle \partial_t \partial_{x_j} v_\nu(t, \cdot) | [\Delta_\nu, T_a^m] \partial_{x_h} w(t, \cdot) \rangle_{L^2} \\ & \leq \frac{1}{N} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + \frac{C_m}{1-s} \|a\|_{\text{Lip}} N \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \end{aligned}$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

This follows from the following lemma whose proof can be found in the appendix.

Lemma 3.13. *Let $m \in \mathbb{N}_0$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$. Then there exists a constant $C_m > 0$ such that, for all $w(t, \cdot) \in \mathcal{H}$,*

$$\sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j} [\Delta_\nu, T_a^m] \partial_{x_h} w(t, \cdot) \right\|_{L^2}^2 \leq \frac{C_m}{1-s} \|a\|_{\text{Lip}}^2 \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2$$

with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

Also the next proposition follows immediately from this lemma.

Proposition 3.14. *Let $m \in \mathbb{N}_{\geq 3}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \text{Lip}(\mathbb{R}_x^n)$ and $s \in (0, 1)$. Then, for $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$ there exists a constant $C_m > 0$ such that for all $w \in H^{1-s-\alpha t}(\mathbb{R}_x^n)$*

$$\begin{aligned} & \sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \nu \langle \partial_{x_j} v_\nu(t, \cdot) | [\Delta_\nu, T_a^m] \partial_{x_h} w(t, \cdot) \rangle_{L^2} \\ & \leq \frac{C_m}{1-s} \|a\|_{\text{Lip}} \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \end{aligned}$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

4. Proof of Proposition 2.8

In order to simplify the presentation, we shall write the proof only for $n = 1$. As already mentioned, one may also include lower-order terms with the appropriate regularity in x ; see Section 2.2. The latter can be handled with the techniques of the present work following the scheme of [6].

To make the proof more readable, we divide it into several steps. First the operator will be transformed by a change of variables involving the weight function, and then we shall introduce the paraproduct and microlocalize the operator. After that, we shall use the estimates of Section 3.2 and conclude the proof for $\nu = 0$ and $\nu \geq 1$ separately. After that, in Section 5 we shall show how the stability estimate follows from the energy estimate.

4.1. Preliminaries - transformation, microlocalization, approximation

Let $u \in \mathcal{H}$ be a solution of the equation

$$Pu = \partial_t u + \partial_x(a(t, x)\partial_x u) = 0$$

on the strip $[0, T] \times \mathbb{R}_x$. In what follows, $\alpha_1 > 0$, $\bar{\lambda} > 1$ and $\bar{\gamma} > 0$ are constants to be determined later. Set $\alpha := \max\{\alpha_1, T^{-1}\}$, take $s \in (0, 1)$, and set $\sigma := \frac{1-s}{\alpha}$, $\tau := \frac{\sigma}{4}$. For $\gamma \geq \bar{\gamma}$, $\lambda \geq \bar{\lambda}$ and $\beta \geq \sigma + \tau$, define $w(t, x) = e^{\gamma t} e^{-\beta \Phi_\lambda(\frac{t+\tau}{\beta})} u(t, x)$. Then w satisfies the following equation:

$$w_t - \gamma w + \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) w + \partial_x(a(t, x)\partial_x w) = 0.$$

Now we add and subtract $\partial_x T_a^m \partial_x w$, with T_a^m as defined in (3.2), and obtain

$$w_t - \gamma w + \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) w + \partial_x(T_a^m \partial_x w) + \partial_x((a - T_a^m)\partial_x w) = 0. \quad (4.1)$$

We set $u_\nu = \Delta_\nu u$, $w_\nu = \Delta_\nu w$ and $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$. The function v_ν satisfies

$$\begin{aligned} \partial_t v_\nu &= \gamma v_\nu - \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu - \partial_x(T_a^m \partial_x v_\nu) - \alpha \log(2) \nu v_\nu \\ &\quad - 2^{-(s+\alpha t)\nu} \partial_x([\Delta_\nu, T_a^m] \partial_x w) - 2^{-(s+\alpha t)\nu} \Delta_\nu \partial_x((a - T_a^m) \partial_x w). \end{aligned} \quad (4.2)$$

Next, we compute the scalar product of (4.2) with $(t + \tau) \partial_t v_\nu$ and obtain

$$\begin{aligned} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 &= \gamma(t + \tau) \langle v_\nu | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) \left\langle \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \right\rangle_{L^2} \\ &\quad - (t + \tau) \langle \partial_x(T_a^m \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - \alpha \log(2) (t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \partial_x([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \Delta_\nu \partial_x((a - T_a^m) \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}. \end{aligned} \quad (4.3)$$

To proceed, we have to regularize the coefficient $a(t, x)$ with respect to t . Therefore, we pick an even, non-negative $\rho \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\rho) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $\int_{\mathbb{R}} \rho(s) ds = 1$. For $\varepsilon \in (0, 1]$, we set

$$a_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} a(s, x) \rho\left(\frac{t-s}{\varepsilon}\right) ds.$$

A straightforward computation shows that for all $\varepsilon \in (0, 1]$, we have

$$a_\varepsilon(t, x) \geq a_0 > 0 \quad (4.4)$$

$$|a_\varepsilon(t, x) - a(t, x)| \leq A_{LL}\varepsilon(|\log(\varepsilon)| + 1) \quad (4.5)$$

as well as

$$|\partial_t a_\varepsilon(t, x)| \leq A_{LL}\|\rho'\|_{L^1(\mathbb{R})}(|\log(\varepsilon)| + 1)$$

for all $(t, x) \in [0, T] \times \mathbb{R}_x$. From these properties of $a_\varepsilon(t, x)$, the fact that $T_{a+b} = T_a + T_b$ and Proposition 3.5, we immediately get

Lemma 4.1. *Let $m \in \mathbb{N}_0$ and $u \in L^2(\mathbb{R}_x^n)$. Then*

$$\|(T_a^m - T_{a_\varepsilon}^m)u\|_{L^2} \leq C_m A_{LL}\varepsilon(|\log(\varepsilon)| + 1)\|u\|_{L^2} \text{ and}$$

$$\|T_{\partial_t a_\varepsilon}^m u\|_{L^2} \leq C_m A_{LL}\|\rho'\|_{L^1(\mathbb{R})}\|u\|_{L^2}$$

hold.

We set

$$a_\nu(t, x) := a_\varepsilon(t, x), \text{ with } \varepsilon = 2^{-2\nu}.$$

We replace T_a^m by $T_{a_\nu}^m + T_a^m - T_{a_\nu}^m$ in the third term of the right hand side of (4.3) and we obtain

$$\begin{aligned} (t + \tau)\|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 &= \gamma(t + \tau) \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) \left\langle \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \right\rangle_{L^2} \\ &\quad - (t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - \alpha \log(2)(t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}. \end{aligned} \quad (4.6)$$

Now we replace $\partial_t v_\nu(t, \cdot)$ in

$$-\alpha \log(2)(t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}$$

by the expression in the right hand side of (4.2) and we obtain

$$\begin{aligned} &-\alpha \log(2)(t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} = \\ &\quad - \alpha \gamma \log(2) \nu (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\ &\quad + \alpha \log(2)(t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\ &\quad + \alpha \log(2)(t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_x T_a^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\ &\quad + \alpha^2 (\log(2))^2 (t + \tau) \nu^2 \|v_\nu(t, \cdot)\|_{L^2}^2 \\ &\quad + \alpha \log(2) \nu 2^{-(s+\alpha t)\nu} (t + \tau) \langle v_\nu(t, \cdot) | \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) \rangle_{L^2} \\ &\quad + \alpha \log(2) \nu 2^{-(s+\alpha t)\nu} (t + \tau) \langle v_\nu(t, \cdot) | \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t, \cdot)) \rangle_{L^2}. \end{aligned} \quad (4.7)$$

With (4.6) and (4.7), we obtain

$$\begin{aligned}
(t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 = & \gamma(t + \tau) \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& + \alpha \log(2)(t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& + \alpha \log(2)(t + \tau) \nu \langle v_\nu(t, \cdot) | \partial_x T_a^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& + \alpha^2 (\log(2))^2 (t + \tau) \nu^2 \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& - \alpha \gamma \log(2)(t + \tau) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& + \alpha \log(2)(t + \tau) \nu 2^{-(s+\alpha t)\nu} \langle v_\nu(t, \cdot) | \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) \rangle_{L^2} \\
& + \alpha \log(2)(t + \tau) \nu 2^{-(s+\alpha t)\nu} \langle v_\nu(t, \cdot) | \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t, \cdot)) \rangle_{L^2} \\
& - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \partial_x ([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \Delta_\nu \partial_x ((a - T_a^m) \partial_x w(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Integration by parts with respect to t yields

$$\gamma(t + \tau) \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} = \frac{\gamma}{2} \frac{d}{dt} \left((t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_\nu(t, \cdot)\|_{L^2}^2$$

and

$$\begin{aligned}
& - (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \langle v_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} = \\
& - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Next we investigate the term $-(t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}$. From (3.2) it can be seen that $\partial_t T_{a_\nu}^m = T_{\partial_t a_\nu}^m + T_a^m \partial_t$. A simple computation shows that

$$\begin{aligned}
& - (t + \tau) \langle \partial_x (T_{a_\nu}^m \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} = \\
& \frac{1}{2} \frac{d}{dt} \left((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t, \cdot) | \partial_x v_\nu(t, \cdot) \rangle_{L^2} \right) \\
& - \frac{1}{2} \langle T_{a_\nu}^m \partial_x v_\nu(t, \cdot) | \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle T_{\partial_t a_\nu}^m \partial_x v_\nu(t, \cdot) | \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle \partial_t \partial_x v_\nu(t, \cdot) | ((T_{a_\nu}^m)^* - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
& (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 = \\
& \frac{\gamma}{2} \frac{d}{dt} ((t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2) - \frac{\gamma}{2} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) \\
& + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& - (t + \tau) \langle \partial_x ((T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot)) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
& + \frac{1}{2} \frac{d}{dt} ((t + \tau) \langle T_{a_\nu}^m \partial_x v_\nu(t, \cdot) | \partial_x v_\nu(t, \cdot) \rangle_{L^2}) \\
& - \frac{1}{2} \langle T_{a_\nu}^m \partial_x v_\nu(t, \cdot) | \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle \partial_x v_\nu(t, \cdot) | T_{a_\nu}^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle \partial_t \partial_x v_\nu(t, \cdot) | ((T_{a_\nu}^m)^* - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& - \alpha \gamma \log(2) (t + \tau) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& + \alpha \log(2) (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& - \alpha \log(2) (t + \tau) \nu \langle \partial_x v_\nu(t, \cdot) | T_a^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& + \alpha^2 (\log(2))^2 (t + \tau) \nu^2 \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& + \alpha \log(2) \nu 2^{-(s+\alpha t)\nu} (t + \tau) \langle v_\nu(t, \cdot) | \mathcal{X}_\nu(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \mathcal{X}_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2},
\end{aligned} \tag{4.8}$$

where we have set

$$\mathcal{X}_\nu(t, \cdot) := (\partial_x ([\Delta_\nu, T_a^m] \partial_x w(t, \cdot)) + \Delta_\nu (\partial_x ((a - T_a^m) \partial_x w(t, \cdot)))) .$$

4.2. Estimates for $\nu = 0$

Setting $\nu = 0$, we get from (4.8)

$$\begin{aligned}
(t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 = & \frac{\gamma}{2} \frac{d}{dt} ((t + \tau) \|v_0(t, \cdot)\|_{L^2}^2) - \frac{\gamma}{2} \|v_0(t, \cdot)\|_{L^2}^2 \\
& - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \right) \\
& + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \\
& - (t + \tau) \langle \partial_x ((T_a^m - T_{a_0}^m) \partial_x v_0(t, \cdot)) | \partial_t v_0(t, \cdot) \rangle_{L^2} \\
& + \frac{1}{2} \frac{d}{dt} ((t + \tau) \langle \partial_x v_0(t, \cdot) | T_{a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2}) \\
& - \frac{1}{2} \langle \partial_x v_0(t, \cdot) | T_{a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2} - \frac{1}{2} (t + \tau) \langle \partial_x v_0(t, \cdot) | T_{\partial_t a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2} \\
& - \frac{1}{2} (t + \tau) \langle ((T_{a_0}^m)^* - T_{a_0}^m) \partial_x v_0(t, \cdot) | \partial_t \partial_x v_0(t, \cdot) \rangle_{L^2} \\
& - (t + \tau) \langle \mathcal{X}_0(t, \cdot) | \partial_t v_0(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Using Propositions 3.1, 3.5 and Lemma 4.1, for $N_1, N_2 > 0$ we get

$$\begin{aligned}
| \langle \partial_x v_0(t, \cdot) | T_{\partial_t a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2} | & \leq C_{a,m}^{(1)} \|v_0\|_{L^2}^2, \\
| \langle T_{a-a_0}^m \partial_x v_0(t, \cdot) | \partial_x \partial_t v_0(t, \cdot) \rangle_{L^2} | & \leq C_{a,m}^{(2)} N_1 \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{N_1} \|\partial_t v_0(t, \cdot)\|_{L^2}^2, \\
| \langle ((T_{a_0}^m)^* - T_{a_0}^m) \partial_x v_0(t, \cdot) | \partial_t \partial_x v_0(t, \cdot) \rangle_{L^2} | & \leq C_{a,m}^{(3)} N_2 \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{N_2} \|\partial_t v_0(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

Now we choose N_1 and N_2 so large that

$$\frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{2} < 0$$

and $\bar{\gamma}$ so large that

$$-\frac{\gamma}{4} + \left(C_{a,m}^{(1)} + C_{a,m}^{(2)} N_1 + C_{a,m}^{(3)} N_2 \right) \left(\frac{7}{8} \sigma + \tau \right) < 0$$

for $\gamma \geq \bar{\gamma}$. Hence, the term

$$\begin{aligned}
& C_{a,m}^{(1)} (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + C_{a,m}^{(2)} N_1 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 \\
& + C_{a,m}^{(3)} N_2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2
\end{aligned}$$

is absorbed by $-\frac{\gamma}{4} \|v_\nu(t, \cdot)\|_{L^2}^2$ and the term

$$\frac{1}{N_1} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{N_2} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2$$

is absorbed by $-\frac{1}{2}(t+\tau)\|\partial_t v_0(t, \cdot)\|_{L^2}^2$. Hence, we get

$$\begin{aligned} & \frac{\gamma}{2} \frac{d}{dt} ((t+\tau)\|v_0(t, \cdot)\|_{L^2}^2) - \frac{\gamma}{4} \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \Phi'_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \\ & - \frac{1}{2} \frac{d}{dt} \left((t+\tau) \Phi'_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \right) + \frac{1}{2} \frac{t+\tau}{\beta} \Phi''_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \\ & + \frac{1}{2} \frac{d}{dt} ((t+\tau) \langle \partial_x v_0(t, \cdot) | T_{a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2}) - (t+\tau) \langle \mathcal{X}_0 | \partial_t v_0(t, \cdot) \rangle_{L^2}. \end{aligned}$$

Further, we recall that Φ fulfills equation (2.4), i.e.

$$y \Phi''_\lambda(y) = -\lambda (\Phi'_\lambda(y))^2 \mu \left(\frac{1}{\Phi'_\lambda(y)} \right) = -\lambda \Phi'_\lambda(y) (1 + |\log \left(\frac{1}{\Phi'_\lambda(y)} \right)|)$$

for $\lambda > 1$. From this, we see that

$$\frac{1}{2} \Phi'_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \frac{t+\tau}{\beta} \Phi''_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 < 0$$

holds, and thus, we get

$$\begin{aligned} & \frac{\gamma}{8} \|v_0(t, \cdot)\|_{L^2}^2 \\ & \leq -\frac{1}{2} (t+\tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt} ((t+\tau)\|v_0(t, \cdot)\|_{L^2}^2) - \frac{\gamma}{8} \|v_0(t, \cdot)\|_{L^2}^2 \\ & + \frac{1}{2} \frac{d}{dt} ((t+\tau) \langle \partial_x v_0(t, \cdot) | T_{a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2}) - (t+\tau) \langle \mathcal{X}_0 | \partial_t v_0(t, \cdot) \rangle_{L^2} \\ & - \frac{1}{2} \frac{d}{dt} \left((t+\tau) \Phi'_\lambda \left(\frac{t+\tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \right). \end{aligned}$$

Using Propositions 3.6 and 3.5 as well as integrating in t over $[0, p] \subseteq [0, \frac{7}{8}\sigma]$, we obtain

$$\begin{aligned} & \frac{\gamma}{8} \int_0^p \|v_0(t, \cdot)\|_{L^2}^2 dt \\ & \leq \left(\frac{\gamma}{2} + C_{m,a}^{(4)} \right) (p+\tau) \|v_0(p, \cdot)\|_{L^2}^2 + \frac{1}{2} \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \|v_0(0, \cdot)\|_{L^2}^2 \\ & - \frac{\gamma}{8} \int_0^p \|v_0(t, \cdot)\|_{L^2}^2 dt - \frac{1}{2} \int_0^p (t+\tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 dt \\ & - \int_0^p (t+\tau) \langle \mathcal{X}_0(t, \cdot) | \partial_t v_0(t, \cdot) \rangle_{L^2} dt, \end{aligned}$$

where we have used

$$|\langle \partial_x v_0(p, \cdot) | T_{a_0}^m \partial_x v_0(p, \cdot) \rangle_{L^2}| \leq C_{m,a}^{(4)} \|v_0(p, \cdot)\|_{L^2}^2$$

and, applying Proposition 3.6,

$$\langle \partial_x v_0(t, \cdot) | T_{a_0}^m \partial_x v_0(t, \cdot) \rangle_{L^2} \geq \frac{\kappa}{2} \|\partial_x v_0(t, \cdot)\|_{L^2}^2$$

choosing m large enough.

4.3. Estimates for $\nu \geq 1$

Now we consider (4.8) for $\nu \geq 1$. From Lemma 4.1, for N_3 and $N_4 > 0$ we obtain

$$\begin{aligned} & | \langle (T_a^m - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot) | \partial_x \partial_t v_\nu(t, \cdot) \rangle_{L^2} | \\ & \leq C_{a,m}^{(5)} N_3 \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{N_3} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2, \end{aligned} \quad (4.9)$$

where we also used the fact that $(\mu(\varepsilon))^2 \leq \mu(\varepsilon)$, $\varepsilon \in (0, 1]$, and

$$| \langle \partial_x v_\nu(t, \cdot) | T_{\partial_t a_\nu}^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} | \leq C_{a,m}^{(6)} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \quad (4.10)$$

as well as

$$\begin{aligned} & | \langle ((T_{a_\nu}^m)^* - T_{a_\nu}^m) \partial_x v_\nu(t, \cdot) | \partial_t \partial_x v_\nu(t, \cdot) \rangle_{L^2} | \\ & \leq C_{a,m}^{(7)} N_4 \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{N_4} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \end{aligned} \quad (4.11)$$

which follows from Proposition 3.7. Using again the positivity estimate in Proposition 3.6, we obtain

$$\begin{aligned} & -\alpha \log(2)(t + \tau) \nu \langle \partial_x v_\nu(t, \cdot) | T_{a_\nu}^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\ & \leq -\alpha C_{a,m}^{(8)} (t + \tau) \nu 2^{2\nu} \|\partial_x v_\nu(t, \cdot)\|_{L^2}^2. \end{aligned} \quad (4.12)$$

Now we choose N_3 and N_4 so large that

$$\frac{1}{N_3} + \frac{1}{N_4} - \frac{1}{2} < 0,$$

and α_1 large enough so that

$$-\frac{\alpha_1}{2} C_{a,m}^{(8)} + N_3 C_{a,m}^{(5)} + C_{a,m}^{(6)} + C_{a,m}^{(7)} N_4 < 0,$$

and we set $\alpha := \max\{T^{-1}, \alpha_1\}$. With this choice, we get

$$\begin{aligned}
& \frac{\gamma}{4} \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{2} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \leq \frac{\gamma}{2} \frac{d}{dt} ((t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2) - \frac{\gamma}{4} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad - \frac{1}{2} \frac{d}{dt} \left((t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) \\
& \quad + \frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \frac{d}{dt} ((t + \tau) \langle \partial_x v_\nu(t, \cdot) | T_{a_\nu}^m \partial_x v_\nu(t, \cdot) \rangle_{L^2}) \\
& \quad - \alpha \gamma \log(2) (t + \tau) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 - \frac{1}{2} \langle \partial_x v_\nu(t, \cdot) | T_{a_\nu}^m \partial_x v_\nu(t, \cdot) \rangle_{L^2} \\
& \quad + \alpha \log(2) (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \alpha^2 (\log(2))^2 \nu^2 \|v_\nu(t, \cdot)\|_{L^2}^2 - \frac{\alpha}{2} C_{a,m}^{(8)} (t + \tau) \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \alpha \log(2) \nu 2^{-(s+\alpha t)\nu} (t + \tau) \langle v_\nu(t, \cdot) | \mathcal{X}_\nu(t, \cdot) \rangle_{L^2} \\
& \quad - (t + \tau) 2^{-(s+\alpha t)\nu} \langle \mathcal{X}_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2}.
\end{aligned} \tag{4.13}$$

Since $y\Phi''_\lambda(y) = -\lambda\Phi'_\lambda(y)(1 + |\log(\Phi'_\lambda(y))|)$, if we take $\lambda \geq \bar{\lambda} > 2$ we have

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \leq -\frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right),$$

and hence the term $\frac{1}{2} \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2$ in (4.13) is absorbed by the term $\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2$. Now we need to absorb

$$\alpha \log(2) (t + \tau) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right) \nu \|v_\nu(t, \cdot)\|_{L^2}^2. \tag{4.14}$$

There are two terms in (4.13) that will help to achieve this. One is

$$-\frac{\alpha}{4} C_{a,m}^{(8)} (t + \tau) \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \tag{4.15}$$

and the other one is

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda \left(\frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2. \tag{4.16}$$

Let $\tilde{C}_{a,m}^{(8)} = \min\{4 \log(2), C_{a,m}^{(8)}\}$. If $\nu \geq (\log(2))^{-1} \log\left(\frac{4 \log(2)}{\tilde{C}_{a,m}^{(8)}}\right) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right)$, then

$$-\frac{C_{a,m}^{(8)}}{4} \alpha \nu 2^{2\nu} \leq -\alpha \log(2) \Phi'_\lambda \left(\frac{t + \tau}{\beta} \right).$$

If on the contrary $\nu < (\log(2))^{-1} \log\left(\frac{4\log(2)}{\tilde{C}_{a,m}^{(8)}}\right) \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)$ then $\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) > 2^\nu$ and, hence, by (2.4), we obtain

$$\begin{aligned}
\frac{1}{4} \frac{t+\tau}{\beta} \Phi''_\lambda\left(\frac{t+\tau}{\beta}\right) &= -\frac{1}{4} \lambda \left(\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\right)^2 \mu\left(\frac{1}{\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right) \\
&\leq -\frac{1}{4} \lambda \left(\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\right)^2 \mu\left(\frac{1}{\frac{4\log(2)}{\tilde{C}_{a,m}^{(8)}} \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right) \\
&\leq -\frac{1}{4} \lambda \frac{\tilde{C}_{a,m}^{(8)}}{4\log(2)} \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) \left(1 + \left|\log\left(\frac{4\log(2)}{\tilde{C}_{a,m}^{(8)}} \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\right)\right|\right) \\
&\leq -\frac{1}{4} \lambda \frac{\tilde{C}_{a,m}^{(8)}}{4\log(2)} \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) (1 + \nu \log(2)) \\
&\leq -\lambda C_{a,m}^{(9)} \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) \nu,
\end{aligned}$$

where we have used the fact that the function μ is increasing. Consequently, if we choose $\lambda \geq \bar{\lambda}$ with

$$\bar{\lambda} \geq \frac{\alpha \log(2) \left(\frac{7}{8}\sigma + \tau\right)}{C_{a,m}^{(9)}},$$

we have

$$\frac{1}{4} \frac{t+\tau}{\beta} \Phi''_\lambda\left(\frac{t+\tau}{\beta}\right) \leq -\alpha \log(2) (t+\tau) \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)$$

and hence, the term (4.14) is compensated by (4.15) and (4.16). Now we consider the term

$$\alpha^2 \log^2(2) \nu^2 \|v_\nu(t, \cdot)\|_{L^2}. \quad (4.17)$$

If $\nu \geq (\log(2))^{-1} \log\left(\frac{\alpha(2\log(2))^2}{C_{a,m}^{(8)}}\right) =: \bar{\nu}_1$ then

$$-\frac{C_{a,m}^{(8)}}{4} \alpha \nu 2^{2\nu} + \alpha^2 \log^2(2) \nu^2 \leq 0.$$

If $\nu \leq \bar{\nu}_1$, we choose $\bar{\gamma}$ possibly larger such that

$$\frac{\gamma}{4} \geq \alpha^2 \log^2(2) \bar{\nu}_1^2 \left(\frac{7}{8}\sigma + \tau\right)$$

for all $\gamma \geq \bar{\gamma}$. We obtain

$$-\frac{\gamma}{4} + \alpha^2 \log^2(2) \nu \leq 0,$$

and consequently (4.17) is absorbed by

$$-\frac{\alpha}{4} C_{a,m}^{(8)} (t+\tau) \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 - \frac{\gamma}{4} \|v_\nu(t, \cdot)\|_{L^2}^2.$$

The term $-\alpha\gamma\log(2)(t+\tau)\nu\|v_\nu(t,\cdot)\|_{L^2}^2$ can be neglected since it is negative. However, we stress here that it is a crucial term in order to achieve our energy estimate for an equation including also lower order terms. Eventually, recalling also Propositions 3.1 and 3.6, we obtain

$$\begin{aligned}
& \frac{1}{2}(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + \frac{\gamma}{8}\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& \leq \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_\nu(t,\cdot)\|_{L^2}^2\right) - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2\right) \\
& \quad + \frac{1}{2}\frac{d}{dt}\left((t+\tau)\langle\partial_x v_\nu(t,\cdot)|T_{a,\nu}^m\partial_x v_\nu(t,\cdot)\rangle_{L^2}\right) \\
& \quad - \frac{\kappa}{8}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& \quad - \frac{\alpha}{2}\log(2)C_{a,m}^{(8)}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& \quad + \alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\rangle_{L^2} \\
& \quad - (t+\tau)2^{-(s+\alpha t)\nu}\langle\mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} - \frac{\gamma}{8}\|v_\nu(t,\cdot)\|_{L^2}^2.
\end{aligned}$$

Integrating over $[0, p] \subseteq [0, \frac{7}{8}\sigma]$, we get

$$\begin{aligned}
& \frac{\kappa}{8}\int_0^p 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 dt + \frac{\gamma}{8}\int_0^p \|v_\nu(t,\cdot)\|_{L^2}^2 dt \\
& \leq \tau\Phi'_\lambda\left(\frac{\tau}{\beta}\right)\|v_\nu(0,\cdot)\|_{L^2}^2 + \left(\frac{\gamma}{2} + C_{a,m}^{(10)}2^{2\nu}\right)(p+\tau)\|v_\nu(p,\cdot)\|_{L^2}^2 \\
& \quad - \frac{\alpha}{2}\log(2)C_{a,m}^{(8)}\int_0^p (t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 dt - \frac{\gamma}{8}\int_0^p \|v_\nu(t,\cdot)\|_{L^2}^2 dt \\
& \quad - \frac{1}{2}\int_0^p (t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 dt \\
& \quad - \int_0^p (t+\tau)2^{-(s+\alpha t)\nu}\langle\mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} dt \\
& \quad + \alpha\log(2)\int_0^p \nu 2^{-(s+\alpha t)\nu}(t+\tau)\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\rangle_{L^2} dt.
\end{aligned}$$

Now we sum over ν and we obtain

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^p \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^p \sum_{\nu \geq 0} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{\nu \geq 0} \|v_\nu(0, \cdot)\|_{L^2}^2 - \frac{\gamma}{8} \int_0^p \sum_{\nu \geq 0} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \frac{1}{2} \int_0^p (t + \tau) \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \frac{\gamma}{2} (p + \tau) \sum_{\nu \geq 0} \|v_\nu(p, \cdot)\|_{L^2}^2 + C_{a,m}^{(10)} (p + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(p, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\alpha}{2} \log(2) C_{a,m}^{(8)} \int_0^p (t + \tau) \sum_{\nu \geq 0} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \int_0^p (t + \tau) \sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \langle \mathcal{X}_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} dt \\
& \quad + \alpha \log(2) \int_0^p (t + \tau) \sum_{\nu \geq 0} \nu 2^{-(s+\alpha t)\nu} \langle v_\nu(t, \cdot) | \mathcal{X}_\nu(t, \cdot) \rangle_{L^2} dt.
\end{aligned}$$

Using the results from Sections 3.3 and 3.4, we have the estimates

$$\begin{aligned}
& - \int_0^p (t + \tau) \sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \langle \mathcal{X}_\nu(t, \cdot) | \partial_t v_\nu(t, \cdot) \rangle_{L^2} dt \\
& \leq \eta \int_0^p (t + \tau) \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \frac{C_{a,m,s}^{(11)}}{\eta} \int_0^p (t + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt
\end{aligned}$$

and

$$\begin{aligned}
& \alpha \log(2) \int_0^p (t + \tau) \sum_{\nu \geq 0} \nu 2^{-(s+\alpha t)\nu} \langle v_\nu(t, \cdot) | \mathcal{X}_\nu(t, \cdot) \rangle_{L^2} dt \\
& \leq \alpha \log(2) C_{a,m,s}^{12} \int_0^p (t + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt.
\end{aligned}$$

4.4. End of the proof

So far we have obtained

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^p \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^p \sum_{\nu \geq 0} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{\nu \geq 0} \|v_\nu(0, \cdot)\|_{L^2}^2 - \frac{\gamma}{8} \int_0^p \sum_{\nu \geq 0} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \frac{\gamma}{2} (p + \tau) \sum_{\nu \geq 0} \|v_\nu(p, \cdot)\|_{L^2}^2 + C_{a,m}^{(10)} (p + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(p, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\alpha}{2} \log(2) C_{a,m}^{(8)} \int_0^p (t + \tau) \sum_{\nu \geq 0} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \frac{1}{2} \int_0^p (t + \tau) \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt + \eta \int_0^p (t + \tau) \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \left(\alpha \log(2) C_{a,m,s}^{12} + \frac{C_{a,m,s}^{(11)}}{\eta} \right) \int_0^p (t + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt.
\end{aligned}$$

Now we take $\eta < \frac{1}{2}$ and choose $\bar{\nu}_2 := \left\lceil \left(\alpha \log(2) C_{a,m,s}^{12} + \frac{C_{a,m,s}^{(11)}}{\eta} \right) \frac{2}{\alpha \log(2) C_{a,m}^{(8)}} \right\rceil$.

With this, we have

$$\begin{aligned}
& -\frac{\alpha}{2} \log(2) C_{a,m}^{(8)} \int_0^p (t + \tau) \sum_{\nu \geq \bar{\nu}_2} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad \left(\alpha \log(2) C_{a,m,s}^{12} + \frac{C_{a,m,s}^{(11)}}{\eta} \right) \int_0^p (t + \tau) \sum_{\nu \geq \bar{\nu}_2} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \leq 0.
\end{aligned}$$

To absorb the remaining parts of the sum, we choose $\bar{\gamma}$ larger (if necessary) such that

$$-\frac{\gamma}{8} + \left(\frac{7}{8} \sigma + \tau \right) \left(\alpha \log(2) C_{a,m,s}^{12} + \frac{C_{a,m,s}^{(11)}}{\eta} \right) 2^{2\bar{\nu}_2} < 0$$

for all $\gamma \geq \bar{\gamma}$. This leads to

$$\begin{aligned}
& -\frac{\gamma}{8} \int_0^p \sum_{\nu < \bar{\nu}_2} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \left(\alpha \log(2) C_{a,m,s}^{12} + \frac{C_{a,m,s}^{(11)}}{\eta} \right) \int_0^p (t + \tau) \sum_{\nu < \bar{\nu}_2} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \leq 0.
\end{aligned}$$

All in all, we finally obtain

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^p \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^p \sum_{\nu \geq 0} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \sum_{\nu \geq 0} \|v_\nu(0, \cdot)\|_{L^2}^2 + \frac{\gamma}{2} (p + \tau) \sum_{\nu \geq 0} \|v_\nu(p, \cdot)\|_{L^2}^2 \\
& \quad + C_{a,m}^{(10)} (p + \tau) \sum_{\nu \geq 0} 2^{2\nu} \|v_\nu(p, \cdot)\|_{L^2}^2.
\end{aligned}$$

From this, going back to u_ν , we have

$$\begin{aligned}
& \frac{\kappa}{8} \int_0^p e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \sum_{\nu \geq 0} 2^{2(1-s-\alpha t)\nu} \|u_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \frac{\gamma}{8} \int_0^p e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \|u_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq C_{a,m}^{(10)} (p + \tau) e^{2\gamma p} e^{-2\beta\Phi_\lambda(\frac{p+\tau}{\beta})} \sum_{\nu \geq 0} 2^{2(1-s-\alpha p)\nu} \|u_\nu(p, \cdot)\|_{L^2}^2 \\
& \quad + \frac{\gamma}{2} (p + \tau) e^{2\gamma p} e^{-2\beta\Phi_\lambda(\frac{p+\tau}{\beta})} \sum_{\nu \geq 0} 2^{-(s+\alpha p)\nu} \|u_\nu(p, \cdot)\|_{L^2}^2 \\
& \quad + \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \sum_{\nu \geq 0} 2^{-2s\nu} \|u_\nu(0, \cdot)\|_{L^2}^2.
\end{aligned}$$

Using Proposition 3.2, the weighted energy estimate (2.5) follows. \square

5. Proof of Theorem 2.4

In this section we show how the stability estimate in Theorem 2.4 follows from the energy estimate in Theorem 2.8. To this end, we need two lemmas whose proof is left to the reader.

Lemma 5.1. *There exists $\gamma_0 > 0$ such that if $\gamma \geq \gamma_0$ then, for every $u \in \mathcal{H}$ solution of (2.1), the function $E(t) = e^{2\gamma t} \|u(t, \cdot)\|_{L^2}^2$ is not decreasing in $[0, T]$.*

The next lemma contains an estimate of the H^1 -norm of a solution of (2.1) by its L^2 -norm. This estimate is crucial in gaining (2.3) from (2.5).

Lemma 5.2. *There exist a constant C such that, if $u \in \mathcal{H}$ be a solution of (2.1) in $[0, T]$, then*

$$\inf_{t \in [\frac{5}{7}\sigma, \frac{7}{8}\sigma]} \|u(t)\|_{H^1}^2 \leq \frac{C}{\sigma} \sup_{t \in [\frac{5}{7}\sigma, \frac{7}{8}\sigma]} \|u(t)\|_{L^2}^2.$$

The constant C depends only on κ .

We start from the inequality

$$\begin{aligned} & \int_0^p e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-s-\alpha t}}^2 dt \\ & \leq M \left[(p+\tau) e^{2\gamma p} e^{-2\beta\Phi_\lambda(\frac{p+\tau}{\beta})} \|u(p, \cdot)\|_{H^{1-s-\alpha p}}^2 \right. \\ & \quad \left. + \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right] \end{aligned}$$

which is valid for $p \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. For every $\sigma^* \in (\frac{5}{8}\sigma, \frac{7}{8}\sigma)$, we have

$$\begin{aligned} & \int_0^{\sigma^*} e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-s-\alpha t}}^2 dt \\ & \leq M \left[(\sigma^* + \tau) e^{2\gamma\sigma^*} e^{-2\beta\Phi_\lambda(\frac{\sigma^*+\tau}{\beta})} \|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 \right. \\ & \quad \left. + \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right], \end{aligned}$$

where $\beta \geq \sigma + \tau$. Now we take $p \in [0, \bar{\sigma}]$ with $\bar{\sigma} = \frac{1}{2}(\frac{\sigma}{2} - \tau) = \frac{\sigma}{8}$, so $2p + \tau \leq 2\bar{\sigma} + \tau = \frac{\sigma}{2} < \frac{5}{8}\sigma < \sigma^*$, and hence

$$\begin{aligned} & \int_p^{2p+\tau} e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-s-\alpha t}}^2 dt \\ & \leq M \left[(\sigma^* + \tau) e^{2\gamma\sigma^*} e^{-2\beta\Phi_\lambda(\frac{\sigma^*+\tau}{\beta})} \|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 \right. \\ & \quad \left. + \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right]. \end{aligned}$$

Since $\frac{1}{8}(1-s) \leq 1-s-\alpha t \leq 1-s$, we have

$$\|u(t, \cdot)\|_{H^{1-s-\alpha t}} \geq \|u(t, \cdot)\|_{L^2}.$$

Hence, with Lemma 5.1,

$$\begin{aligned} & e^{2\gamma p} (p+\tau) \|u(p, \cdot)\|_{L^2}^2 e^{-2\beta\Phi_\lambda(\frac{2p+\tau}{\beta})} \\ & \leq M \left[(\sigma^* + \tau) e^{2\gamma\sigma^*} e^{-2\beta\Phi_\lambda(\frac{\sigma^*+\tau}{\beta})} \|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 \right. \\ & \quad \left. + \tau \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right]. \end{aligned}$$

Since $\Phi'_\lambda \geq 1$, we have

$$\begin{aligned}
\|u(p, \cdot)\|_{L^2}^2 &\leq M \frac{\sigma^* + \tau}{\tau} e^{2\gamma\sigma^*} \Phi'_\lambda \left(\frac{\tau}{\beta} \right) \\
&\quad \times \left[e^{2\beta\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - 2\beta\Phi_\lambda(\frac{\sigma^*+\tau}{\beta})} \|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 \right. \\
&\quad \left. + e^{2\beta\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - 2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right] \\
&\leq \tilde{M} \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{2\beta\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - 2\beta\Phi'_\lambda(\frac{\sigma^*+\tau}{\beta})} \\
&\quad \times \left[\|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + e^{-\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right].
\end{aligned}$$

Now we have $\frac{\sigma^*+\tau}{\beta} \geq \frac{5}{8}\frac{\sigma+\tau}{\beta}$, which implies

$$\Phi_\lambda \left(\frac{\sigma^* + \tau}{\beta} \right) \geq \Phi_\lambda \left(\frac{5}{8}\frac{\sigma + \tau}{\beta} \right)$$

and hence

$$\begin{aligned}
\|u(p, \cdot)\|_{L^2}^2 &\leq \tilde{M} \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{2\beta\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - 2\beta\Phi_\lambda(\frac{5\sigma/8+\tau}{\beta})} \\
&\quad \times \left[\|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + e^{-\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right].
\end{aligned}$$

By the concavity of Φ_λ , we have

$$\begin{aligned}
&2\beta\Phi_\lambda \left(\frac{\sigma/2+\tau}{\beta} \right) - 2\beta\Phi_\lambda \left(\frac{5\sigma/8+\tau}{\beta} \right) \\
&\leq \Phi'_\lambda \left(\frac{5\sigma/8+\tau}{\beta} \right) \left(\frac{\sigma/2+\tau}{\beta} - \frac{5\sigma/8+\tau}{\beta} \right) = -\Phi'_\lambda \left(\frac{5\sigma/8+\tau}{\beta} \right) \frac{\sigma}{8\beta}.
\end{aligned}$$

This implies

$$\begin{aligned}
\|u(p, \cdot)\|_{L^2}^2 &\leq \tilde{M} \Phi'_\lambda \left(\frac{\tau}{\beta} \right) e^{-\frac{\sigma}{8}\Phi'_\lambda(\frac{5\sigma/8+\tau}{\beta})} \\
&\quad \times \left(\|u(\sigma^*, \cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right).
\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}
\Phi'_\lambda \left(\frac{5\sigma/8+\tau}{\beta} \right) &= \Psi_\lambda \left(\frac{5\sigma/8+\tau}{\tau} \frac{\tau}{\beta} \right) \\
&= \exp \left(\left(\frac{5\sigma/8+\tau}{\tau} \right)^{-\lambda} - 1 \right) \Psi_\lambda \left(\frac{\tau}{\beta} \right)^{\left(\frac{5\sigma/8+\tau}{\tau} \right)^{-\lambda}}.
\end{aligned}$$

Setting $\tilde{\delta} := \left(\frac{5\sigma/8+\tau}{\tau} \right)^{-\lambda}$, we have

$$\|u(p, \cdot)\|_{L^2}^2 \leq \tilde{M} \psi_\lambda \left(\frac{\tau}{\beta} \right) e^{-\tilde{N}\psi_\lambda(\frac{\tau}{\beta})^{\tilde{\delta}}} \left(\|u(\sigma^*, \cdot)\|_{H^1}^2 + e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{H^{-s}}^2 \right).$$

Now we choose β such that

$$e^{-\beta\Phi_\lambda(\frac{\tau}{\beta})} = \|u(0, \cdot)\|_{H^{-s}}^{-1}$$

that is

$$\beta = \tau\Lambda^{-1} \left(\frac{1}{\tau} \log \|u(0, \cdot)\|_{L^2} \right).$$

Then there exists $\bar{\rho} > 0$ such that, if $\|u(0, \cdot)\|_{L^2} \leq \bar{\rho}$, then $\beta \geq \sigma + \tau$. With this choice and thanks to Lemma 2.7, we get

$$\|u(p, \cdot)\|_{L^2}^2 \leq \tilde{M} \exp \left(-\tilde{N} \left[\frac{1}{\tau} \log (\|u(0, \cdot)\|_{H^{-s}}) \right]^{\tilde{\delta}} \right) (\|u(\sigma^*, \cdot)\|_{H^1}^2 + 1)$$

for all $\sigma^* \in [\frac{5}{8}\sigma, \frac{7}{8}\sigma]$ and for all $p \in [0, \frac{\sigma}{8}]$. By Lemma 5.2, we finally get

$$\|u(p, \cdot)\|_{L^2}^2 \leq C e^{-\tilde{N}[\frac{1}{\tau} \log (\|u(0, \cdot)\|_{H^{-s}})]^{\tilde{\delta}}} \left(\max_{t \in [\frac{5}{8}\sigma, \frac{7}{8}\sigma]} \|u(t, \cdot)\|_{L^2}^2 + 1 \right).$$

This completes the proof of Theorem 2.4. \square

Appendix

Proof of Proposition 3.10. To estimate

$$\sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \langle \partial_{x_i} \partial_t v_\nu(t, \cdot) | \Delta_\nu((a - T_a^m) \partial_{x_j} w(t, \cdot)) \rangle_{L^2},$$

we introduce a second microlocalization: Setting $w(t, \cdot) = \sum_{\mu \geq 0} w_\mu(t, \cdot)$ and $w_\mu(t, \cdot) = 2^{(s+\alpha t)\nu} v_\mu(t, \cdot)$ (see Section 4.1) we obtain, using Proposition 3.1, that

$$\begin{aligned} & \sum_{\nu \geq 0} 2^{-(s+\alpha t)\nu} \langle \partial_{x_i} \partial_t v_\nu(t, \cdot) | \Delta_\nu((a - T_a^m) \partial_{x_j} w(t, \cdot)) \rangle_{L^2} \\ &= \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)\nu} \langle \partial_{x_i} \partial_t v_\nu(t, \cdot) | \Delta_\nu((a - T_a^m) \partial_{x_j} w_\mu(t, \cdot)) \rangle_{L^2} \\ &= \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} \langle \partial_{x_i} \partial_t v_\nu(t, \cdot) | \Delta_\nu((a - T_a^m) \partial_{x_j} v_\mu(t, \cdot)) \rangle_{L^2} \\ &\leq \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu((a - T_a^m) \partial_{x_j} v_\mu(t, \cdot))\|_{L^2} \\ &\leq \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t, \cdot)\|_{L^2} \\ &\quad + \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_2 \partial_{x_j} v_\mu(t, \cdot)\|_{L^2}. \end{aligned}$$

Since $w(t, \cdot) \in H^1(\mathbb{R}_x^n)$ we have $\partial_x v_\mu \in H^{-s}(\mathbb{R}_x^n)$ and, taking an $s' \in (0, s)$, also $\partial_x v_\mu(t, \cdot) \in H^{-s'}(\mathbb{R}_x^n)$. By Lemma 3.8, we then get

$$\|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t, \cdot)\|_{L^2} \leq C c_\nu^{(\mu)} 2^{-(1-s')\nu} 2^\mu \|v_\mu(t, \cdot)\|_{L^2}$$

and therefore

$$\begin{aligned} & \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t, \cdot)\|_{L^2} \\ & \leq C \sum_{\nu \geq 0} \sum_{\mu \leq \nu} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} c_\nu^{(\mu)} 2^{-(1-s')\nu} 2^\mu 2^{-s'\mu} \|v_\mu(t, \cdot)\|_{L^2} \\ & \leq C \sum_{\nu \geq 0} \sum_{\mu \leq \nu} 2^{-s\alpha t\nu} 2^{s\alpha t\mu} \left(2^{(s'-s)(\nu-\mu)} \|\partial_t v_\nu(t, \cdot)\|_{L^2} \right) \left(c_\nu^{(\mu)} 2^\mu \|v_\mu(t, \cdot)\|_{L^2} \right) \\ & \leq \frac{1}{N} \sum_{\nu \geq 0} \sum_{\mu \leq \nu} 2^{-2(s-s')(\nu-\mu)} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + CN \sum_{\nu \geq 0} \sum_{\mu \leq \nu} (c_\nu^{(\mu)})^2 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2 \\ & \leq \frac{1}{N} \sum_{\nu \geq 0} \left(\sum_{\mu \leq \nu} 2^{2(s-s')\mu} \right) 2^{-2(s-s')\nu} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\ & \quad + CN \sum_{\mu \geq 0} \left(\sum_{\nu \geq 0} (c_\nu^{(\mu)})^2 \right) 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2 \\ & = \frac{1}{N} \sum_{\nu \geq 0} \frac{2^{2(s-s')(\nu+1)} - 1}{2^{2(s-s')\nu} - 1} 2^{-2(s-s')\nu} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + CN \sum_{\mu \geq 0} 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2 \\ & \leq \frac{1}{N} \frac{2^{2(s-s')}}{2^{2(s-s')\nu} - 1} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + CN \sum_{\mu \geq 0} 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2. \end{aligned}$$

By the summation formula of the geometric sum and the integral criterion, we obtain

$$\frac{2^{2(s-s')}}{2^{2(s-s')\nu} - 1} \leq \frac{2^{2(s-s')}}{2^{2(s-s')}(1 - 2^{-2(s-s')})} \leq \frac{C}{s - s'}$$

and, hence,

$$\begin{aligned} & \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t, \cdot)\|_{L^2} \\ & \leq \frac{1}{N} \frac{C}{s - s'} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + CN \sum_{\mu \geq 0} 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2. \end{aligned}$$

On the other hand, we have from Lemma 3.9 that

$$\|\Delta_\nu \Omega_2 \partial_x v_\mu(t, \cdot)\|_{L^2} \leq C \tilde{c}_\nu^{(\mu)} \|v_\mu(t, \cdot)\|_{L^2},$$

and therefore we get

$$\begin{aligned}
& \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_2 \partial_x v_\mu(t, \cdot)\|_{L^2} \\
& \leq C \sum_{\nu \geq 0} \sum_{\mu \geq \nu-4} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_2 \partial_x v_\mu(t, \cdot)\|_{L^2} \\
& \leq C \sum_{\nu \geq 0} \sum_{\mu \geq \nu-4} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \tilde{c}_\nu^{(\mu)} 2^{-\mu} 2^\mu \|v_\mu(t, \cdot)\|_{L^2} \\
& \leq C \sum_{\nu \geq 0} \sum_{\mu \geq \nu-4} 2^{(1-s-\alpha t)\nu} 2^{-(1-s-\alpha t)\mu} \tilde{c}_\nu^{(\mu)} \|\partial_t v_\nu(t, \cdot)\|_{L^2} 2^\mu \|v_\mu(t, \cdot)\|_{L^2} \\
& \leq \frac{1}{N} \sum_{\nu \geq 0} \sum_{\mu \geq \nu-4} 2^{2(1-s-\alpha t)\nu} 2^{-2(1-s-\alpha t)\mu} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + CN \sum_{\nu \geq 0} \sum_{\mu \geq \nu-4} (c_\nu^{(\mu)})^2 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2 \\
& \leq \frac{1}{N} \frac{2^{8(1-s-\alpha t)}}{1 - 2^{-2(1-s-\alpha t)}} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + CN \sum_{\mu \geq 0} \left(\sum_{\nu \leq \mu+4} (c_\nu^{(\mu)})^2 \right) 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

Since $t \in [0, \frac{7}{8}\sigma]$, where $\sigma := \frac{1-s}{\alpha}$, we have $\frac{1}{8}(1-s) \leq 1-s-\alpha t \leq 1-s$ and hence

$$\frac{2^{8(1-s-\alpha t)}}{1 - 2^{-2(1-s-\alpha t)}} \leq \frac{C}{1-s-\alpha t} \leq \frac{C}{1-s}.$$

From that, we get

$$\begin{aligned}
& \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t, \cdot)\|_{L^2} \|\Delta_\nu \Omega_2 \partial_x v_\mu(t, \cdot)\|_{L^2} \\
& \leq \frac{1}{N} \frac{C}{1-s} \sum_{\nu \geq 0} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 + CN \sum_{\mu \geq 0} 2^{2\mu} \|v_\mu(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

This concludes the proof of the proposition. \square

Proof of Lemma 3.13. The proof is very similar to that of [7][Prop. 3.7]. We detail it for the reader's convenience. We have

$$[\Delta_\nu, T_a^m]w = [\Delta_\nu, S_{m-1}a]S_{m+1}w + \sum_{k \geq m+2} [\Delta_\nu, S_{k-3}a]\Delta_k w$$

and get

$$\begin{aligned}
\partial_{x_j} [\Delta_\nu, T_a^m] \partial_{x_h} w &= \partial_{x_j} ([\Delta_\nu, S_{m-1}a] S_{m+1}(\partial_{x_h} w)) \\
&\quad + \partial_{x_j} \left(\sum_{k \geq m+2} [\Delta_\nu, S_{k-3}a] \Delta_k(\partial_{x_h} w) \right)
\end{aligned}$$

since Δ_ν and Δ_k commute and we therefore have that

$$\begin{aligned} [\Delta_\nu, S_{m-1}a]S_{m+1}w &= \Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}aS_{m+1}(\Delta_\nu w) \\ &= \Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}a\Delta_\nu(S_{m+1}w). \end{aligned}$$

This holds analogously for $[\Delta_\nu, S_{k-3}a]\Delta_k w$. Let us consider

$$\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w)) = \partial_{x_j}([\Delta_\nu, S_{m-1}a]\partial_{x_h}(S_{m+1}w)).$$

Looking at the spectrum of this term we see that the term equals to 0 if $\nu \geq m+4$. Moreover, the spectrum is contained in $\{|\xi| \leq 2^{m+3}\}$. From Bernstein's inequality, we have that

$$\|\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w))\|_{L^2} \leq 2^{m+3} \|[\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w)\|_{L^2}.$$

From the well known result of Coifman and Meyer [3, Th. 35], which essentially say that

$$\|[\Delta_\nu, b]\partial_x w\|_{L^2} \leq C \|\nabla_x b\|_{L^\infty} \|w\|_{L^2}, \quad (5.1)$$

where $b \in \text{Lip}(\mathbb{R}_x^n)$ and $w \in H^1(\mathbb{R}_x^n)$, we get

$$\|[\Delta_\nu, S_{m-1}a]\partial_{x_h}(S_{m+1}w)\|_{L^2} \leq C \|a\|_{\text{Lip}} \|S_{m+1}w\|_{L^2}.$$

Further, we have

$$\|S_{m+1}w\|_{L^2} \leq \sum_{k \leq m+1} \|\Delta_k w\|_{L^2} \leq C \sum_{k \leq m+1} 2^{-(1-s-\alpha t)} \varepsilon_k,$$

where $\{\varepsilon_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$ with $\|\{\varepsilon_k\}_k\|_{l^2} \approx \|w\|_{H^{1-s-\alpha t}}$. Using now Hölder's inequality, we obtain

$$\|S_{m+1}w\|_{L^2} \leq C \left(\sum_{k \geq 0} 2^{-2(1-s-\alpha t)} \right)^{\frac{1}{2}} \|w\|_{H^{1-s-\alpha t}} \leq \frac{C}{1-s} \|w\|_{H^{1-s-\alpha t}},$$

where we used the summation formula for the geometric sum as well as the assumption that $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Consequently,

$$\|\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w))\|_{L^2} \leq \frac{C}{1-s} \|a\|_{\text{Lip}} \|w\|_{H^{1-s-\alpha t}}$$

and,

$$\begin{aligned} &\sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j} [\Delta_\nu, S_{k-3}a] \Delta_k (\partial_{x_h} w) \right\|_{L^2}^2 \\ &= \sum_{0 \leq \nu \leq m+3} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j} [\Delta_\nu, S_{k-3}a] \Delta_k (\partial_{x_h} w) \right\|_{L^2}^2 \leq \frac{C_m}{(1-s)^2} \|a\|_{\text{Lip}}^2 \|w\|_{H^{1-s-\alpha t}}^2. \end{aligned} \quad (5.2)$$

Now, we consider

$$\partial_{x_j} \left(\sum_{k \geq m+2} [\Delta_\nu, S_{k-3}a] \Delta_k(\partial_{x_h} w) \right) = \partial_{x_j} \left(\sum_{k \geq m+2} [\Delta_\nu, S_{k-3}a] \partial_{x_h}(\Delta_k w) \right).$$

Looking at $\text{spec}([\Delta_\nu, S_{k-3}a] \Delta_k(\partial_{x_h} w))$, we see that $[\Delta_\nu, S_{k-3}a] \Delta_k(\partial_{x_h} w)$ is identically 0 if $|k - \nu| \geq 4$. This means that the sum runs over at most seven terms: from $\partial_{x_j} [\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)$ up to $\partial_{x_j} [\Delta_\nu, S_\nu a] \partial_{x_h}(\Delta_{\nu+3} w)$, where each of them has a spectrum contained in a ball $\{|\xi| \leq C2^\nu\}$. We consider only one of these terms, e.g. $\partial_{x_j} [\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)$ since the estimates for the others follow analogously. From Bernstein's inequality we get

$$\|\partial_{x_j} [\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)\|_{L^2} \leq C2^\nu \|[\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)\|_{L^2}$$

and, using again (5.1),

$$\|[\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)\|_{L^2} \leq C \|a\|_{\text{Lip}} \|\Delta_\nu w\|_{L^2}.$$

Hence, we have

$$\|\partial_{x_j} [\Delta_\nu, S_{\nu-6}a] \partial_{x_h}(\Delta_{\nu-3} w)\|_{L^2} \leq C2^\nu \|a\|_{\text{Lip}} \|\Delta_\nu w\|_{L^2}.$$

Thus squaring, multiplying by $2^{-2(s+\alpha t)\nu}$ and summing over ν , we get

$$\begin{aligned} \sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \|\partial_{x_j} [\Delta_\nu, S_{\nu-3}] \partial_{x_h}(\Delta_\nu u)\|_{L^2}^2 \\ \leq C \|a\|_{\text{Lip}}^2 \sum_{\nu \geq 0} 2^{2(1-s-\alpha t)\nu} \|\Delta_\nu w\|_{L^2}^2. \end{aligned}$$

With $w \in H^{1-s-\alpha t}(\mathbb{R}_x^n)$ and using Proposition 3.3, we finally get

$$\sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \|\partial_{x_j} [\Delta_\nu, S_{\nu-3}] \partial_{x_h}(\Delta_\nu u)\|_{L^2}^2 \leq C \|a\|_{\text{Lip}}^2 \|u\|_{H^{1-s-\alpha t}}^2.$$

As already mentioned, the other terms can be treated the same way. We finally get

$$\sum_{\nu \geq 0} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j} \left(\sum_{k \geq m+2} [\Delta_\nu, S_{k-3}] \partial_{x_h}(\Delta_k u) \right) \right\|_{L^2}^2 \leq C \|a\|_{\text{Lip}}^2 \|u\|_{H^{1-s-\alpha t}}^2 \quad (5.3)$$

which, putting (5.2) and (5.3) together and using the notation $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$, concludes the proof of the proposition. \square

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